Uniformly monotone functions - definition, properties, characterizations

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Abstract

Quasi-concave functions play an important role in economics and finance as utility functions, measures of risk or other objects used, mainly, in portfolio selection analysis. A special attention is paid to these functions in the minimax theory. Unfortunately, their limited application is due to the fact that supremum, sum, product of quasi-concave functions are typically not quasi-concave. This difficulty is overcome by establishing of uniformly quasi-concave functions, due to Prékopa, Yoda and Subasi (2011). We contribute with a new characterization of uniformly quasi-concave functions that allows easier verification and provide more straightforward insight.

Keywords: Quasi-concave function, uniformly quasi-concave functions, uniformly monotone functions, partial ordering, total ordering, monotonicity.
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1 Introduction

Let us open our discussion with a definition of quasi-concave real function.

Definition 1 *Let $E \subset \mathbb{R}^n$. We say that $f : E \to \mathbb{R}$ is quasi-concave if*

1. *E is convex.*
2. *For each $\Delta \in \mathbb{R}^*$ the level set $\mathsf{lev}_{\geq\alpha} f = \{x \in E : f(x) \succeq \alpha\}$ is convex.*

Alternatively, one can deal with quasi-convex functions; i.e. $-f$ is quasi-concave. All these functions are useful in economics and finance, they serve as utility functions, measures of risk or other objects, mainly in portfolio selection analysis; see [1], [2], [3], [4], [5], [8], [9], [10], [11], [12]. In this paper we focus on quasi-concave functions. Unfortunately, their limited application is due to the fact that supremum, sum, product of quasi-concave functions are typically not quasi-concave. This difficulty is overcome by establishing of uniformly quasi-concave functions, due to Prékopa, Yoda and Subasi (2011), see [7].

Definition 2 *Let $E \subset \mathbb{R}^n$. Then, we say that functions $f_i : E \to \mathbb{R}$, $i = 1, 2, \ldots, m$ are*
uniformly quasi-concave if

1. *E is convex.*
2. *For each $i = 1, 2, \ldots, m$ the function f_i is quasi-concave.*

3. *For each $x, y \in E$ either*

$$\forall i = 1, 2, \ldots, m \quad \min\{f_i(x), f_i(y)\} = f_i(x)$$

or

$$\forall i = 1, 2, \ldots, m \quad \min\{f_i(x), f_i(y)\} = f_i(y).$$

We present a generalization and equivalent descriptions of uniformly quasi-concave functions. This paper improves our considerations published in [6].

2 Introductory definitions

We will deal with functions defined on a common nonempty set E and having values in a totally ordered set $\mathcal{Q} = (\mathcal{Q}, \prec_{\mathcal{Q}})$, e.g. the $\mathcal{Q}$ could be the extended real line $\mathbb{R}^* = [-\infty, +\infty]$ equipped with the natural ordering. On $\mathcal{Q}$, we will employ derived relations $\succ_{\mathcal{Q}}, \preceq_{\mathcal{Q}}, \succeq_{\mathcal{Q}}, \min_{\mathcal{Q}}, \max_{\mathcal{Q}}$. The set of all such functions will be denoted $\mathcal{F}(E, \mathcal{Q})$.

Let us start with definitions of the main objects of our interest.

Definition 3 *We say that a nonempty family $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ is uniformly monotone if for each $x, y \in E$ either*

$$\forall f \in \mathcal{F} \quad \min_{\mathcal{Q}}\{f(x), f(y)\} = f(x)$$

or

$$\forall f \in \mathcal{F} \quad \min_{\mathcal{Q}}\{f(x), f(y)\} = f(y).$$

We are searching for equivalent descriptions of uniformly monotone families of functions.

A characterization is based on level sets. Let us recall appropriate definitions.

Definition 4 *For a function $f : E \to \mathcal{Q}$ and a given level $\Delta \in \mathcal{Q}$ we consider level sets*

$$\begin{aligned}
\mathsf{lev}_{\preceq_{\mathcal{Q}}\Delta} f &= \{x \in E \,:\, f(x) \preceq_{\mathcal{Q}} \Delta\}, \\
\mathsf{lev}_{\prec_{\mathcal{Q}}\Delta} f &= \{x \in E \,:\, f(x) \prec_{\mathcal{Q}} \Delta\}, \\
\mathsf{lev}_{\succeq_{\mathcal{Q}}\Delta} f &= \{x \in E \,:\, f(x) \succeq_{\mathcal{Q}} \Delta\}, \\
\mathsf{lev}_{\succ_{\mathcal{Q}}\Delta} f &= \{x \in E \,:\, f(x) \succ_{\mathcal{Q}} \Delta\}.
\end{aligned}$$

For a nonempty family $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ we determine the sets of all its level sets

$$\begin{aligned}
\mathsf{LEVELs}_{\preceq_{\mathcal{Q}}}(\mathcal{F}) &= \{\mathsf{lev}_{\preceq_{\mathcal{Q}}\Delta} f \,:\, f \in \mathcal{F},\, \Delta \in \mathcal{Q}\}, \\
\mathsf{LEVELs}_{\prec_{\mathcal{Q}}}(\mathcal{F}) &= \{\mathsf{lev}_{\prec_{\mathcal{Q}}\Delta} f \,:\, f \in \mathcal{F},\, \Delta \in \mathcal{Q}\}, \\
\mathsf{LEVELs}_{\succeq_{\mathcal{Q}}}(\mathcal{F}) &= \{\mathsf{lev}_{\succeq_{\mathcal{Q}}\Delta} f \,:\, f \in \mathcal{F},\, \Delta \in \mathcal{Q}\}, \\
\mathsf{LEVELs}_{\succ_{\mathcal{Q}}}(\mathcal{F}) &= \{\mathsf{lev}_{\succ_{\mathcal{Q}}\Delta} f \,:\, f \in \mathcal{F},\, \Delta \in \mathcal{Q}\}.
\end{aligned}$$

Working in a vector space $\mathcal{V}$, convex sets are well-defined and we can correctly define quasi-concave functions.

Definition 5 *Let $\mathcal{V}$ be a vector space and $E \subset \mathcal{V}$ be nonempty. We say that $f \in \mathcal{F}(E, \mathcal{Q})$ is quasi-concave if*

- *E is convex.*
- *For each $\Delta \in \mathcal{Q}$ the level set $\mathsf{lev}_{\succeq_{\mathcal{Q}} \Delta} f$ is convex.*

The definition of uniformly quasi-concave functions introduced in [7] can be also generalized for our setting.

Definition 6 *Let $\mathcal{V}$ be a vector space and $E \subset \mathcal{V}$ be nonempty. We say that a nonempty family $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ is uniformly quasi-concave if*

- *E is convex,*
- *each function of $\mathcal{F}$ is quasi-concave,*
- *$\mathcal{F}$ is uniformly monotone.*

3 Descriptions of uniformly monotone functions

3.1 Partial ordering induced by functions

Any set of functions determines a partial ordering on their common domain. This observation allows us to derive an equivalent characterizations.

Definition 7 *A nonempty family $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ determines a partial ordering $\prec^{\mathcal{F}}$ and an equivalence $\sim^{\mathcal{F}}$ on E by*

$$\begin{array}{lcl} x \prec^{\mathcal{F}} y & \Longleftrightarrow & \begin{array}{l} \forall f \in \mathcal{F} : f(x) \preceq_{\mathcal{Q}} f(y), \\ \exists g \in \mathcal{F} \text{ s.t. } g(x) \prec_{\mathcal{Q}} g(y), \end{array} \\ x \sim^{\mathcal{F}} y & \Longleftrightarrow & \forall f \in \mathcal{F} : f(x) = f(y). \end{array}$$

The partial ordering is giving an equivalent description of uniform monotonicity.

Theorem 1 *Let $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ be a nonempty family. Then $\mathcal{F}$ is uniformly monotone iff the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}} /_{\sim^{\mathcal{F}}}$, i.e. for each couple $x, y \in E$ just one from the three following relations holds*

$$x \prec^{\mathcal{F}} y, \; x \sim^{\mathcal{F}} y, \; y \prec^{\mathcal{F}} x.$$

Proof: We will prove the equivalence.

1. Let $\mathcal{F}$ be uniformly monotone.
   Fix $x, y \in E$, $x \not\sim^{\mathcal{F}} y$.
   Then, there is a function $g \in \mathcal{F}$ s.t. $g(x) \neq g(y)$.
   We have to distinguish two possibilities:
   (a) Let $g(x) \prec_{\mathcal{Q}} g(y)$.
       Hence from uniform monotonicity $\forall f \in \mathcal{F} : f(x) \preceq_{\mathcal{Q}} f(y)$
       Consequently, $x \prec^{\mathcal{F}} y$.

(b) Let $g(x) \succ_{\mathcal{Q}} g(y)$.
Hence from uniform monotonicity $\forall f \in \mathcal{F} : f(x) \succeq_{\mathcal{Q}} f(y)$
Consequently, $y \prec^{\mathcal{F}} x$.

We have proved that the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}} /_{\sim^{\mathcal{F}}}$.

2. Let the factor space $E/_{\sim^{\mathcal{F}}}$ be totally ordered by $\prec^{\mathcal{F}} /_{\sim^{\mathcal{F}}}$.

   Fix $x, y \in E$.

   Since the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}} /_{\sim^{\mathcal{F}}}$, we have to distinguish three possibilities:

   (a) If $x \sim^{\mathcal{F}} y$ then, $\forall f \in \mathcal{F} : f(x) = f(y)$.
   Hence, $\forall f \in \mathcal{F} : \min_{\mathcal{Q}}\{f(x), f(y)\} = f(x) = f(y)$.

   (b) If $x \prec^{\mathcal{F}} y$ then, $\forall f \in \mathcal{F} : f(x) \preceq_{\mathcal{Q}} f(y)$.
   Hence, $\forall f \in \mathcal{F} : \min_{\mathcal{Q}}\{f(x), f(y)\} = f(x)$.

   (c) If $y \prec^{\mathcal{F}} x$ then, $\forall f \in \mathcal{F} : f(x) \succeq_{\mathcal{Q}} f(y)$.
   Hence, $\forall f \in \mathcal{F} : \min_{\mathcal{Q}}\{f(x), f(y)\} = f(y)$.

   We have shown that $\mathcal{F}$ is uniformly monotone.

Q.E.D.

3.2 The set of all level sets

Another equivalent characterization can be received using the set of all level sets.

Theorem 2 *Let $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ be a nonempty family. Then $\mathcal{F}$ is uniformly monotone.*

iff

$\mathsf{LEVELs}_{\preceq_{\mathcal{Q}}}(\mathcal{F})$ *is totally ordered by natural set-ordering.*

iff

$\mathsf{LEVELs}_{\prec_{\mathcal{Q}}}(\mathcal{F})$ *is totally ordered by natural set-ordering.*

iff

$\mathsf{LEVELs}_{\succeq_{\mathcal{Q}}}(\mathcal{F})$ *is totally ordered by natural set-ordering.*

iff

$\mathsf{LEVELs}_{\succ_{\mathcal{Q}}}(\mathcal{F})$ *is totally ordered by natural set-ordering.*

Proof: It is sufficient to prove the equivalence for $\mathsf{LEVELs}_{\preceq_{\mathcal{Q}}}(\mathcal{F})$, since a type of level sets is totally ordered if and only if the other types of level sets are totally ordered. That is because $\mathsf{LEVELs}_{\succ_{\mathcal{Q}}}(\mathcal{F}) = E \setminus \mathsf{LEVELs}_{\preceq_{\mathcal{Q}}}(\mathcal{F})$ and $\mathsf{LEVELs}_{\prec_{\mathcal{Q}}}(\mathcal{F}) = \mathsf{LEVELs}_{\succ_{\mathcal{Q}}^{\pitchfork}}(\mathcal{F})$, $\mathsf{LEVELs}_{\succeq_{\mathcal{Q}}}(\mathcal{F}) = \mathsf{LEVELs}_{\preceq_{\mathcal{Q}}^{\pitchfork}}(\mathcal{F})$, where $\prec_{\mathcal{Q}}^{\pitchfork}$ denotes the reverse ordering to $\prec_{\mathcal{Q}}$.

1. Let $\mathcal{F}$ be uniformly monotone.

   Let $A, B \in \mathsf{LEVELs}_{\preceq_\mathcal{Q}}(\mathcal{F})$ and $A \setminus B \neq \emptyset$.

   Then, $A = \mathsf{lev}_{\preceq_\mathcal{Q}\alpha} f$, $B = \mathsf{lev}_{\preceq_\mathcal{Q}\beta} g$ for some $f, g \in \mathcal{F}$ and $\alpha, \beta \in \mathcal{Q}$.

   Moreover, there is $\xi \in E$ such that $\xi \in A$ and $\xi \notin B$, i.e. $f(\xi) \preceq_\mathcal{Q} \alpha$ and $g(\xi) \succ_\mathcal{Q} \beta$.

   Take $x \in B$; i.e. $g(x) \preceq_\mathcal{Q} \beta$.

   Then, $g(x) \prec_\mathcal{Q} g(\xi)$. Accordingly to uniform monotonicity, $f(x) \preceq_\mathcal{Q} f(\xi)$.

   Thus, $f(x) \preceq_\mathcal{Q} \alpha$ and, consequently, $x \in A$.

   We have derived $A \supset B$. That means that $\mathsf{LEVELs}_{\preceq_\mathcal{Q}}(\mathcal{F})$ is totally ordered by set-ordering.

2. Let $\mathsf{LEVELs}_{\preceq_\mathcal{Q}}(\mathcal{F})$ be totally ordered by set-ordering.

   Take $x, y \in E$.

   Assume $f, g \in \mathcal{F}$ such that $f(x) \prec_\mathcal{Q} f(y)$ and $g(x) \succ_\mathcal{Q} g(y)$.

   Denoting $\alpha = f(x)$, $\beta = g(y)$, we observe

   $x \in \mathsf{lev}_{\preceq_\mathcal{Q}\alpha} f$, $y \notin \mathsf{lev}_{\preceq_\mathcal{Q}\alpha} f$,

   $x \notin \mathsf{lev}_{\preceq_\mathcal{Q}\beta} g$, $y \in \mathsf{lev}_{\preceq_\mathcal{Q}\beta} g$.

   Therefore, $\mathsf{lev}_{\preceq_\mathcal{Q}\alpha} f \neq \mathsf{lev}_{\preceq_\mathcal{Q}\beta} g$, $\mathsf{lev}_{\preceq_\mathcal{Q}\alpha} f \not\subset \mathsf{lev}_{\preceq_\mathcal{Q}\beta} g$, $\mathsf{lev}_{\preceq_\mathcal{Q}\alpha} f \not\supset \mathsf{lev}_{\preceq_\mathcal{Q}\beta} g$, which is in contradiction with the assumption that $\mathsf{LEVELs}_{\preceq_\mathcal{Q}}(\mathcal{F})$ is totally ordered by set-ordering.

   We derive that the factor space $E/_{\sim^\mathcal{F}}$ is totally ordered by $\prec^\mathcal{F} /_{\sim^\mathcal{F}}$.

   Accordingly to Theorem 1, we have shown that $\mathcal{F}$ is uniformly monotone.

Q.E.D.

3.3 Composition of functions

Characterization by means of total ordering of level sets implies characterization using composition of appropriate functions.

Theorem 3 *Let $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ be a nonempty family. Then $\mathcal{F}$ is uniformly monotone iff there is a totally ordered space $\mathcal{X}$, a function $\psi : E \to \mathcal{X}$ and nondecreasing functions $\varphi_f : \mathcal{X} \to \mathcal{Q}$, $f \in \mathcal{F}$ such that for each $f \in \mathcal{F}$ we have a decomposition $f = \varphi_f \circ \psi$.*

Proof:

1. Assume for each $f \in \mathcal{F}$ a decomposition $f = \varphi_f \circ \psi$, where $\psi : E \to \mathcal{X}$, $\varphi_f : \mathcal{X} \to \mathcal{Q}$ is nondecreasing, $\mathcal{X}$ is a totally ordered space.

   Let $x, y \in E$ and $g \in \mathcal{F}$ be with $g(x) \prec_\mathcal{Q} g(y)$. Then $\psi(x) \prec_\mathcal{X} \psi(y)$, since φ_g is nondecreasing.

   For each $f \in \mathcal{F}$, φ_f is nondecreasing, therefore,

   $$f(x) = \varphi_f \circ \psi(x) \preceq_\mathcal{Q} \varphi_f \circ \psi(y) = f(y).$$

   Thus we have shown, the family $\mathcal{F}$ is uniformly monotone.

2. Let $\mathcal{F}$ be uniformly monotone.

   For each $x \in E$, we denote

$$\begin{aligned} \psi(x) &= \{y \in E \,:\, y \preceq_{\mathcal{Q}} x\} = \bigcap_{f \in \mathcal{F}} \mathsf{lev}_{\preceq_{\mathcal{Q}} f(x)} f, \\ \mathcal{X} &= \{\psi(x) \,:\, x \in E\}. \end{aligned}$$

   Immediately, we have $\mathcal{X}$ is totally ordered by set-ordering and $\psi : E \to \mathcal{X}$. Moreover for each $x \in E$, any function $f \in \mathcal{F}$ reaches its maximum on $\psi(x)$ in the point x. It is because $x \in \psi(x)$ and $\psi(x) \subset \mathsf{lev}_{\preceq_{\mathcal{Q}} f(x)} f$.

   Therefore, we can correctly define

$$\varphi_f(G) \;=\; \max_{\mathcal{Q}} \{f(u) \,:\, u \in G\} \quad \text{for each} \quad G \in \mathcal{X},\; f \in \mathcal{F}.$$

   Hence, for each $f \in \mathcal{F}$, $\varphi_f : \mathcal{X} \to \mathcal{Q}$ is nondecreasing and

$$\varphi_f \circ \psi(x) \;=\; \max_{\mathcal{Q}} \{f(\xi) \,:\, \xi \in \psi(x)\} = f(x).$$

Q.E.D.

We see from the proof that it is sufficient if the outer functions in the decomposition are determined on the image of the inner function. Under a slight restriction these functions can be enlarged to the whole ordered space.

Proposition 1 *Let $\mathcal{X}$, $\mathcal{Q}$ be totally ordered spaces and each nonempty subset of $\mathcal{Q}$ possess a supremum and an infimum. Let $D \subset \mathcal{X}$ and $\varphi : D \to \mathcal{Q}$ be nondecreasing function.*

Then, the function φ can be enlarged to a nondecreasing function $\tilde{\varphi} : \mathcal{X} \to \mathcal{Q}$, i.e. $\tilde{\varphi}(d) = \varphi(d)$ for all $d \in D$.

Proof: We extend the function φ to the whole $\mathcal{X}$. For each $t \in \mathcal{X}$, we set

$$\begin{aligned} \tilde{\varphi}(t) &= \sup_{\mathcal{Q}} \{\varphi(d) \,:\, d \preceq_{\mathcal{X}} t,\; d \in D\} \quad \text{if} \quad \exists d \in D \text{ s.t. } d \preceq_{\mathcal{X}} t, \\ &= \inf_{\mathcal{Q}} \{\varphi(d) \,:\, d \in D\} \quad \text{if} \quad \forall d \in D :\; d \succ_{\mathcal{X}} t. \end{aligned}$$

The function $\tilde{\varphi}$ is nondecreasing and $\tilde{\varphi}(d) = \varphi(d)$ for all $d \in D$.

Q.E.D.

4 Particular cases

Immediate task is if the inner totally ordered space could be taken as the real line. In this section we present three particular cases where it is the case and one counter example.

4.1 Finite and countable families

Theorem 4 *For a nonempty finite family $\mathcal{F} \subset \mathcal{F}(E, \mathbb{R}^*)$ the following statements are equivalent:*

1. *$\mathcal{F}$ is uniformly monotone.*

2. *There are a function* $\psi : E \to \mathbb{R}$ *and nondecreasing functions* $\varphi_f : \psi(E) \to \mathbb{R}^*$ *for each* $f \in \mathcal{F}$ *such that* $f = \varphi_f \circ \psi$ *for all* $f \in \mathcal{F}$.

3. *There are a function* $\psi : E \to \mathbb{R}$ *and nondecreasing functions* $\varphi_f : \mathbb{R} \to \mathbb{R}^*$ *for each* $f \in \mathcal{F}$ *such that* $f = \varphi_f \circ \psi$ *for all* $f \in \mathcal{F}$.

Proof: Considered family possesses finite number of members, say $m \in \mathbb{N}$. Let us index its members as $\mathcal{F} = \{f_1, f_2, \dots, f_m\}$.

To prove announced equivalences, we prove step by step chain of implications.

1. Evidently, (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

2. Let functions of $\mathcal{F}$ be uniformly monotone.

   We take an increasing bijection $\chi : \mathbb{R}^* \to [0, 1]$, e.g. $\chi(t) = \frac{\arctan(t)}{\pi} + \frac{1}{2}$, and set

   $\psi = \chi \circ f_1 + \chi \circ f_2 + \dots + \chi \circ f_m$.

   Hence,

$$\begin{aligned} x \prec^{\mathcal{F}} y &\iff \psi(x) < \psi(y), \\ x \sim^{\mathcal{F}} y &\iff \psi(x) = \psi(y), \\ y \prec^{\mathcal{F}} x &\iff \psi(x) > \psi(y). \end{aligned}$$

   For $i = 1, 2, \dots, m$: we define $\varphi_i : \psi(E) \to \mathbb{R}^*$ such that for $d \in \psi(E)$ we set

$$\varphi_i(d) = f_i(x) \iff \psi(x) = d.$$

   The definition is correct because of

$$\psi(x) = \psi(y) \iff x \sim^{\mathcal{F}} y \iff \forall i = 1, 2, \dots, m : \ f_i(x) = f_i(y).$$

   We have constructed a function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_i : \psi(E) \to \mathbb{R}^*$, $i = 1, 2, \dots, m$ such that $f_i = \varphi_i \circ \psi$ for all $i = 1, 2, \dots, m$.

   Thus, statement (2) is fulfilled.

3. Accordingly to Proposition 1 for each $i = 1, 2, \dots, m$, we extend the function φ_i to the whole $\mathbb{R}$ as nondecreasing function.

   Denoting $\tilde{\varphi}_i$ the extension, we have constructed a function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\tilde{\varphi}_i : \mathbb{R} \to \mathbb{R}^*$, $i = 1, 2, \dots, m$ such that $f_i = \tilde{\varphi}_i \circ \psi$ for all $i = 1, 2, \dots, m$.

   Thus, statement (3) is fulfilled.

Q.E.D.

Similar ideas work for a countable family, also.

Theorem 5 *For a countable family* $\mathcal{F} \subset \mathcal{F}(E, \mathbb{R}^*)$ *the following statements are equivalent:*

1. *$\mathcal{F}$ is uniformly monotone.*
2. *There are a function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_f : \psi(E) \to \mathbb{R}^*$ for each $f \in \mathcal{F}$ such that $f = \varphi_f \circ \psi$ for all $f \in \mathcal{F}$.*
3. *There are a function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_f : \mathbb{R} \to \mathbb{R}^*$ for each $f \in \mathcal{F}$ such that $f = \varphi_f \circ \psi$ for all $f \in \mathcal{F}$.*

Proof: Considered family possesses countable number of members. Let us index its members as $\mathcal{F} = \{f_i, i \in \mathbb{N}\}$.

To prove announced equivalences, we prove step by step chain of implications.

1. Evidently, (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).
2. Let functions of $\mathcal{F}$ be uniformly monotone.

   We take an increasing bijection $\chi : \mathbb{R}^* \to [0, 1]$, e.g. $\chi(t) = \frac{\arctan(t)}{\pi} + \frac{1}{2}$, and set

   $\psi = \sum_{i=1}^{+\infty} 2^{-i} \chi \circ f_i$.

   Hence,

   $$\begin{array}{lcl} x \prec^{\mathcal{F}} y & \Longleftrightarrow & \psi(x) < \psi(y), \\ x \sim^{\mathcal{F}} y & \Longleftrightarrow & \psi(x) = \psi(y), \\ y \prec^{\mathcal{F}} x & \Longleftrightarrow & \psi(x) > \psi(y). \end{array}$$

   For $i \in \mathbb{N}$ we define $\varphi_i : \psi(E) \to \mathbb{R}^*$ such that for $d \in \psi(E)$ we set

   $$\varphi_i(d) = f_i(x) \quad \Longleftrightarrow \quad \psi(x) = d.$$

   The definition is correct because of

   $$\psi(x) = \psi(y) \Longleftrightarrow x \sim^{\mathcal{F}} y \Longleftrightarrow \forall i \in \mathbb{N} : \; f_i(x) = f_i(y).$$

   We have constructed a function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_i : \psi(E) \to \mathbb{R}^*$, $i \in \mathbb{N}$ such that $f_i = \varphi_i \circ \psi$ for all $i \in \mathbb{N}$.

   Thus, statement (2) is fulfilled.
3. Accordingly to Proposition 1 for each $i \in \mathbb{N}$, we extend the function φ_i to the whole $\mathbb{R}$ as nondecreasing function.

   Denoting $\tilde{\varphi}_i$ the extension, we have constructed a function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\tilde{\varphi}_i : \mathbb{R} \to \mathbb{R}^*$, $i \in \mathbb{N}$ such that $f_i = \tilde{\varphi}_i \circ \psi$ for all $i \in \mathbb{N}$.

   Thus, statement (3) is fulfilled.

Q.E.D.

4.2 Topological arguments

Employing topology, we can receive an interesting result.

Definition 8 *If $\mathcal{T}$ is a topological space we denote*

$$\mathcal{RF}(\mathcal{T}) = \{A \subset \mathcal{T} \,:\, \mathrm{clo}(\mathrm{int}(A)) = A\}$$

the set of all regular closed sets.

Lemma 1 *Let $\mathcal{T}$ be a topological space and $A, B \in \mathcal{RF}(\mathcal{T})$. If $A \setminus B \neq \emptyset$ then there is an open set G such that $G \neq \emptyset$ and $G \subset A \setminus B$.*

Proof: Assuming $\mathrm{int}(A) \setminus B = \emptyset$ we have $\mathrm{int}(A) \subset B$ which leads to $A = \mathrm{clo}(\mathrm{int}(A)) \subset \mathrm{clo}(B) = B$ that contradicts the assumption $A \setminus B \neq \emptyset$.

B is a closed set and therefore it is sufficient to set $G = \mathrm{int}(A) \setminus B$.

Q.E.D.

Theorem 6 *Let $\mathcal{T}$ be a topological space possessing a finite Borel measure μ with property $\mu(G) > 0$ for all open sets $G \subset \mathcal{T}$. Let $E \subset \mathcal{T}$ be closed, $E \neq \emptyset$ and $\mathcal{F} \subset \mathcal{F}(E, \mathbb{R}^*)$, $\mathcal{F} \neq \emptyset$. If for all $x \in E$ we have $\{y \in E \,:\, y \preceq_{\mathcal{Q}} x\} \in \mathcal{RF}(\mathcal{T})$ the following statements are equivalent:*

1. *$\mathcal{F}$ is uniformly monotone.*
2. *There are a function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_f : \psi(E) \to \mathbb{R}^*$ for each $f \in \mathcal{F}$ such that $f = \varphi_f \circ \psi$ for all $f \in \mathcal{F}$.*
3. *There are a function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_f : \mathbb{R} \to \mathbb{R}^*$ for each $f \in \mathcal{F}$ such that $f = \varphi_f \circ \psi$ for all $f \in \mathcal{F}$.*

Proof: It is sufficient to show (1) $\Rightarrow$ (2). The rest of proof follows from Theorems 2, 3 and Proposition 1.

Therefore, we assume $\mathcal{F}$ is uniformly monotone. According to Theorem 2, the set $\mathsf{LEVELs}_{\preceq}(\mathcal{F})$ is totally ordered. Now, we repeat construction from the proof of Theorem 3. For each $x \in E$, we set

$$\begin{aligned} \psi(x) &= \{y \in E \,:\, y \preceq_{\mathcal{Q}} x\} = \bigcap_{f \in \mathcal{F}} \mathsf{lev}_{\preceq_{\mathcal{Q}} f(x)} f, \\ \mathcal{X} &= \{\psi(x) \,:\, x \in E\}. \end{aligned}$$

The assumption implies $\mathcal{X} \subset \mathcal{RF}(\mathcal{T})$. According to Lemma 1 and properties of μ, function $\rho : \mathcal{X} \to \mathbb{R} : L \to \mu(L)$ is increasing 1-1-mapping.

Setting $\tilde{\psi} = \rho \circ \psi$, $\tilde{\varphi}_f = \varphi_f \circ \rho^{-1}$, we are receiving description required in the theorem, i.e. $\tilde{\psi} : E \to \mathbb{R}$ and nondecreasing functions $\tilde{\varphi}_f : \tilde{\psi}(E) \to \mathbb{R}^*$ for each $f \in \mathcal{F}$ such that $f = \tilde{\varphi}_f \circ \tilde{\psi}$ for all $f \in \mathcal{F}$.

That concludes the proof.

Q.E.D.

4.3 Counter example

Accepting Axiom of Choice the set of all reals can be well-ordered, say $(\mathbb{R}, \prec)$, see Zermelo's theorem. Recall properties of well-ordering

- $(\mathbb{R}, \prec)$ is totally ordered.
- Each subset of reals possesses a minimal member in $\prec$.

For each $r \in \mathbb{R}$ we define a function $f_r : \mathbb{R} \to \{0, 1\}$ such that $f_r(x) = 0$ for $x \prec r$, $f_r(r) = 0$ and $f_r(x) = 1$ for $x \succ r$.

The counter example is done because the family $\mathcal{F} = \{f_r : r \in \mathbb{R}\}$ is uniformly monotone since $\mathsf{LEVELs}_{\prec}(\mathcal{F})$ is totally ordered by set-inclusion, but, cannot be imbedded into $\mathbb{R}$ equipped with natural ordering.

Therefore, we cannot arrange any inner function with values in reals.

5 Uniformly quasi-concave functions

The observations can be summarized to give equivalent description for uniformly quasi-concave functions. List of general descriptions looks like.

Theorem 7 *Let $\mathcal{V}$ be a vector space, $E \subset \mathcal{V}$, $E \neq \emptyset$ and $\mathcal{F} \subset \mathcal{F}(E, \mathcal{Q})$ be nonempty. Then the following statements are equivalent:*

1. *$\mathcal{F}$ is uniformly quasi-concave.*
2. *$\mathcal{F}$ is uniformly monotone and consists of quasi-concave functions.*
3. *Each function of $\mathcal{F}$ is quasi-concave and the factor space $E/_{\sim^{\mathcal{F}}}$ is totally ordered by $\prec^{\mathcal{F}}/_{\sim^{\mathcal{F}}}$.*
4. *$\mathsf{LEVELs}_{\succeq}(\mathcal{F})$ is totally ordered by set-ordering and consists of convex sets.*
5. *There is a totally ordered space $\mathcal{X}$, a quasi-concave function $\psi : E \to \mathcal{X}$ and nondecreasing functions $\varphi_f : \mathcal{X} \to \mathcal{Q}$ for each $f \in \mathcal{F}$ such that $f = \varphi_f \circ \psi$ for each $f \in \mathcal{F}$.*

Proof: The observation combines Theorems 1, 2 and 3 together with an observation that constructions in their proofs are preserving quasi-concave functions and convex sets.

Q.E.D.

Now, we consider cases possessing characterizations with inner functions leading to reals.

Theorem 8 *Let $\mathcal{V}$ be a vector space, $E \subset \mathcal{V}$, $E \neq \emptyset$ and $\mathcal{F} \subset \mathcal{F}(E, \mathbb{R}^*)$ be nonempty and at most countable family. Then the following statements are equivalent:*

1. *$\mathcal{F}$ is uniformly quasi-concave.*
2. *There are a quasi-concave function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_f : \psi(E) \to \mathbb{R}^*$, $f \in \mathcal{F}$ such that for all $f \in \mathcal{F}$ we have $f = \varphi_f \circ \psi$.*

3. *There are a quasi-concave function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_f : \mathbb{R} \to \mathbb{R}^*$, $f \in \mathcal{F}$ such that for all $f \in \mathcal{F}$ we have $f = \varphi_f \circ \psi$.*

Proof: Theorem combines Theorems 4, 5 and observation that the constructions in their proofs preserve quasi-concave functions because of sum of uniformly quasi-concave functions is quasi-concave, for proof see Prékopa, Yoda and Subasi (2011).

Q.E.D.

Theorem 9 *Let $\mathcal{V}$ be a topological vector space possessing a finite Borel measure μ with property $\mu(G) > 0$ for all open sets $G \subset \mathcal{T}$. Let $E \subset \mathcal{V}$ be closed nonempty set and $\mathcal{F} \subset \mathcal{F}(E, \mathbb{R}^*)$ be nonempty. If for all $x \in E$ we have $\{y \in E : y \preceq_{\mathcal{Q}} x\} \in \mathcal{RF}(\mathcal{T})$ the following statements are equivalent:*

1. *$\mathcal{F}$ is uniformly quasi-concave.*
2. *There are a quasi-concave function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_f : \psi(E) \to \mathbb{R}^*$, $f \in \mathcal{F}$ such that for all $f \in \mathcal{F}$ we have $f = \varphi_f \circ \psi$.*
3. *There are a quasi-concave function $\psi : E \to \mathbb{R}$ and nondecreasing functions $\varphi_f : \mathbb{R} \to \mathbb{R}^*$, $f \in \mathcal{F}$ such that for all $f \in \mathcal{F}$ we have $f = \varphi_f \circ \psi$.*

Proof: Theorem combines Theorems 3, 6 and observation that the constructions in their proofs preserve quasi-concave functions.

Q.E.D.

6 Examples

As an example we consider a family of Gaussian curves and, then, we proceed to the example presented in [7].

We will work in a finite dimension $d \in \mathbb{N}$, denoting by $\mathbb{R}_+$ positive reals, by $\mathbb{R}_{+,0}$ non-negative reals, by $\mathsf{PDM}\,(d)$ the set of all positive definite matrices of type $d \times d$, by $\Lambda\,(\Sigma)$ the largest eigenvalue of a matrix $\Sigma \in \mathsf{PDM}\,(d)$.

Let us begin with two observations from linear algebra.

Lemma 2 *Let $x, y \in \mathbb{R}^d$ be linearly independent and $b \in \mathbb{R}^2$. Then, there is $h \in \mathbb{R}^d$ such that $\langle\, x, h\,\rangle = b_1$, $\langle\, y, h\,\rangle = b_2$.*

Proof: Consider matrix $A = (x\,y)$. The equation $A^\top z = b$ possesses a solution, since rank of A is 2.

Q.E.D.

Lemma 3 *Let $\Sigma, \Gamma \in \mathsf{PDM}\,(d)$. If for each $x \in \mathbb{R}^d$ the vectors Σx, Γx are linearly dependent, then, there is $\alpha > 0$ such that $\Sigma = \alpha\Gamma$.*

Proof:

1. Take $x \in \mathbb{R}^d$, $x \neq 0$.

   Set $\alpha = \frac{\langle\, x, \Gamma x\,\rangle}{\langle\, x, \Sigma x\,\rangle}$, then, $\alpha > 0$ and $\Gamma x = \alpha \Sigma x$, since Σx, Γx are linearly dependent and both matrices are positively definite.

2. Matrix Σ is positively definite, then, there are $\xi_1, \xi_2, ..., \xi_d \in \mathbb{R}^d$ linearly independent eigenvectors. Each ξ_i corresponds to an eigenvalue $\lambda_i > 0$.

   Take $x \in \mathbb{R}^d$, $x \neq 0$.

   Then, one has an expression $x = \sum_{i=1}^{d} c_i \xi_i$ with $c_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, d\}$ properly chosen.

   There are $\alpha_i > 0$, $i \in \{1, 2, \ldots, d\}$ and $\gamma > 0$ such that

$$\Gamma x = \gamma \Sigma x = \gamma \Sigma \left(\sum_{i=1}^{d} c_i \xi_i \right) = \sum_{i=1}^{d} \gamma c_i \Sigma \xi_i = \sum_{i=1}^{d} \gamma c_i \lambda_i \xi_i,$$
$$\Gamma \xi_i = \alpha_i \Sigma \xi_i = \alpha_i \lambda_i \xi_i \text{ for all } i \in \{1, 2, \ldots, d\}.$$

   Consequently,

$$0 = \Gamma x - \sum_{i=1}^{d} c_i \Gamma \xi_i = \sum_{i=1}^{d} (\gamma - \alpha_i) c_i \lambda_i \xi_i.$$

   We know that $\xi_1, \xi_2, ..., \xi_d$ are linearly independent and all eigenvalues of Σ are positive. Therefore, for all $i \in \{1, 2, \ldots, d\}$, $c_i \neq 0$ we have $\gamma = \alpha_i$.

We have proved, there is $\alpha > 0$ such that $\Gamma = \alpha \Sigma$.

Q.E.D.

Now, we recall notion of a differentiable function.

Definition 9 *Let $G \subset \mathbb{R}^d$ be open set, $f : G \to \mathbb{R}$ and $x \in G$. We call f to be differentiable at x, whenever, gradient $\nabla f(x)$ exists and there is a function $\varphi : G - x \to \mathbb{R}$ vanishing at the origin, i.e. $\lim_{z \to 0} \varphi(z) = 0$, such that*

$$f(y) - f(x) \quad = \quad \langle \nabla f(x), y - x \rangle + \|y - x\| \varphi(y - x) \quad \forall y \in G.$$

Lemma 4 *Let $G \subset \mathbb{R}^d$ be open set, $f, g : G \to \mathbb{R}$ and $x \in G$. If f, g are differentiable at x and $\nabla f(x)$, $\nabla g(x)$ are linearly independent then there is a direction $h \in \mathbb{R}^d$ and $\delta > 0$ such that $f(x) < f(x + th)$ and $g(x) > g(x + th)$ for all $t \in (0, \delta)$.*

Hence, if f, g are contained in a family of real functions then the family cannot be uniformly monotone.

Proof: Functions f, g are differentiable at x. Therefore, there are functions $\varphi, \psi : G - x \to \mathbb{R}$ vanishing at the origin, such that for all $y \in G$:

$$\begin{aligned} f(y) - f(x) &= \langle \nabla f(x), y - x \rangle + \|y - x\| \varphi(y - x), \\ g(y) - g(x) &= \langle \nabla g(x), y - x \rangle + \|y - x\| \psi(y - x). \end{aligned}$$

Gradients of considered functions are linearly independent. According to Lemma 2, there is a direction $h \in \mathbb{R}^d$ such that

$$\langle \nabla f(x), h \rangle = 1, \ \langle \nabla g(x), h \rangle = -1.$$

Hence, there is $\delta > 0$ with property $\|h\| |\varphi(th)| < \frac{1}{2}$, $\|h\| |\psi(th)| < \frac{1}{2}$ for all $t \in (0, \delta)$.

$$\begin{aligned} f(x + th) - f(x) &= t \langle \nabla f(x), h \rangle + t \|h\| \varphi(th) = t(1 + \|h\| \varphi(th)) > \frac{t}{2} > 0, \\ g(x + th) - g(x) &= t \langle \nabla g(x), h \rangle + t \|h\| \psi(th) = t(-1 + \|h\| \psi(th)) < -\frac{t}{2} < 0. \end{aligned}$$

Q.E.D.

6.1 Gaussian curves

Let us denote

$$\upsilon\left(\bullet|\alpha,\mu,\Sigma\right):\mathbb{R}^d\to\mathbb{R}:x\in\mathbb{R}^d\mapsto\alpha\exp\left\{-\left\langle\, x-\mu,\Sigma^{-1}(x-\mu)\,\right\rangle\right\},$$

whenever $\alpha>0$, $\mu\in\mathbb{R}^d$, $\Sigma\in\mathsf{PDM}\left(d\right)$.

Lemma 5 *Let $\mathcal{M}\subset\mathbb{R}_+\times\mathbb{R}^d\times\mathsf{PDM}\left(d\right)$. Then, a family of curves $\Upsilon_{\mathcal{M}}=\left\{\upsilon\left(\bullet|\alpha,\mu,\Sigma\right):(\alpha,\mu,\Sigma)\in\mathcal{M}\right\}$ is uniformly monotone iff there are $m\in\mathbb{R}^d$ and $\mathbf{V}\in\mathsf{PDM}\left(d\right)$ such that for all $(\alpha,\mu,\Sigma)\in\mathcal{M}$ we have $\mu=m$, $\Sigma=\frac{\Lambda(\Sigma)}{\Lambda(\mathbf{V})}\mathbf{V}$.*

Proof: Take $(\alpha,\mu,\Sigma),(\beta,\nu,\Gamma)\in\mathcal{M}$.

1. Assume $\mu\neq\nu$.

   Then,

   $$\upsilon\left(\mu|\alpha,\mu,\Sigma\right)>\upsilon\left(\nu|\alpha,\mu,\Sigma\right),\ \upsilon\left(\mu|\alpha,\nu,\Sigma\right)<\upsilon\left(\nu|\alpha,\nu,\Sigma\right).$$

   Therefore, family $\Upsilon_{\mathcal{M}}$ is not uniformly monotone.

2. Let $\mu=\nu$.

   For any $a>0$, $b>0$ the function $f:\mathbb{R}\to\mathbb{R}:t\in\mathbb{R}\mapsto a\exp\{bt\}$ is increasing.

   Therefore, it remains to compare two functions

   $$\kappa_1:\mathbb{R}^d\to\mathbb{R}:x\in\mathbb{R}^d\mapsto-\left\langle\, x-\mu,\left(\frac{1}{\Lambda\left(\Sigma\right)}\Sigma\right)^{-1}(x-\mu)\,\right\rangle,$$
   $$\kappa_2:\mathbb{R}^d\to\mathbb{R}:x\in\mathbb{R}^d\mapsto-\left\langle\, x-\mu,\left(\frac{1}{\Lambda\left(\Gamma\right)}\Gamma\right)^{-1}(x-\mu)\,\right\rangle.$$

   Functions are quadratic. Particularly, they are continuously differentiable with gradients

   $$\nabla\kappa_1\left(x\right)=-2\left(\frac{1}{\Lambda\left(\Sigma\right)}\Sigma\right)^{-1}(x-\mu),$$
   $$\nabla\kappa_2\left(x\right)=-2\left(\frac{1}{\Lambda\left(\Gamma\right)}\Gamma\right)^{-1}(x-\mu).$$

   (a) Let $x\in\mathbb{R}^d$ such that $\nabla\kappa_1\left(x\right)$, $\nabla\kappa_2\left(x\right)$ be linearly independent.
   According to lemma 4, the family $\Upsilon_{\mathcal{M}}$ is not uniformly monotone.

   (b) Let $\nabla\kappa_1\left(x\right)$, $\nabla\kappa_2\left(x\right)$ be linearly dependent for all $x\in\mathbb{R}^d$.
   According to lemma 3, there is $\alpha>0$ with property
   $\left(\frac{1}{\Lambda(\Gamma)}\Gamma\right)^{-1}=\alpha\left(\frac{1}{\Lambda(\Sigma)}\Sigma\right)^{-1}$.
   Then, $\frac{1}{\Lambda(\Gamma)}\Gamma=\frac{1}{\alpha\Lambda(\Sigma)}\Sigma$.
   Comparing largest eigenvalues of matrices, we receive $\alpha=1$ and $\Gamma=\frac{\Lambda(\Gamma)}{\Lambda(\Sigma)}\Sigma$.

Proposition of Lemma 5 is shown.

Q.E.D.

In accordance with Theorem 3, we derived family $\Upsilon_{\mathcal{M}}$ is uniformly monotone iff its functions can be expressed like $\upsilon\left(\bullet|\alpha,\mu,\Sigma\right)=\phi_{\alpha,\Sigma}\circ\psi$. where

$$\begin{aligned}
&\phi_{\alpha,\Sigma}:\mathbb{R}\to\mathbb{R}:t\in\mathbb{R}\mapsto\alpha\exp\left\{\frac{\Lambda\left(\mathbf{V}\right)}{\Lambda\left(\Sigma\right)}t\right\},\\
&\psi:\mathbb{R}^d\to\mathbb{R}:x\in\mathbb{R}^d\mapsto-\left\langle\, x-m,\mathbf{V}^{-1}(x-m)\,\right\rangle.
\end{aligned}$$

6.2 Second example

Let us denote $\mathsf{E}_d=\mathbb{R}^d\setminus\{0\}$ and define functions

$$\kappa\left(\bullet|\alpha,\mu,\Sigma\right):\mathsf{E}_d\to\mathbb{R}:x\in\mathsf{E}_d\mapsto\frac{\alpha-\langle\, x,\mu\,\rangle}{\sqrt{\langle\, x,\Sigma x\,\rangle}},$$

whenever $\alpha>0$, $\mu\in\mathbb{R}^d$, $\Sigma\in\mathsf{PDM}\left(d\right)$.

Lemma 6 *Let $\mathcal{M}\subset\mathbb{R}_+\times\mathbb{R}^d\times\mathsf{PDM}\left(d\right)$. Then, a family of curves $\mathcal{K}_{\mathcal{M}}=\left\{\kappa\left(\bullet|\alpha,\mu,\Sigma\right):(\alpha,\mu,\Sigma)\in\mathcal{M}\right\}$ is uniformly monotone iff there are $m\in\mathbb{R}^d$ and $\mathbf{V}\in\mathsf{PDM}\left(d\right)$ such that for all $(\alpha,\mu,\Sigma)\in\mathcal{M}$ we have $\mu=\alpha m$, $\Sigma=\frac{\Lambda(\Sigma)}{\Lambda(\mathbf{V})}\mathbf{V}$.*

Proof:

1. Evidently $\kappa\left(\bullet|\alpha,\mu,\Sigma\right)=\frac{\alpha}{\sqrt{\Lambda(\Sigma)}}\kappa\left(\bullet|1,\frac{1}{\alpha}\mu,\frac{1}{\Lambda(\Sigma)}\Sigma\right)$, thus, $\mathcal{K}_{\mathcal{M}}$ is uniformly monotone iff $\widetilde{\mathcal{K}}_{\mathcal{M}}=\left\{\kappa\left(\bullet|1,\frac{1}{\alpha}\mu,\frac{1}{\Lambda(\Sigma)}\Sigma\right):(\alpha,\mu,\Sigma)\in\mathcal{M}\right\}$ is uniformly monotone.

2. Take, $\mu,\nu\in\mathbb{R}^d$ and $\Sigma,\Gamma\in\mathsf{PDM}\left(d\right)$ with $\Lambda\left(\Sigma\right)=\Lambda\left(\Gamma\right)=1$.

   Let us abbreviate $\kappa_1(x)=\kappa\left(x|1,\mu,\Sigma\right)$, $\kappa_2(x)=\kappa\left(x|1,\nu,\Gamma\right)$.

   Considered functions are continuously differentiable on E_d with gradient

$$\begin{aligned}
\nabla\kappa_1(x)&=-\frac{1}{\sqrt{\langle\, x,\Sigma x\,\rangle}}\mu-\frac{1-\langle\, x,\mu\,\rangle}{\langle\, x,\Sigma x\,\rangle^{3/2}}\Sigma x,\\
\nabla\kappa_2(x)&=-\frac{1}{\sqrt{\langle\, x,\Gamma x\,\rangle}}\nu-\frac{1-\langle\, x,\nu\,\rangle}{\langle\, x,\Gamma x\,\rangle^{3/2}}\Gamma x.
\end{aligned}$$

   (a) Let μ, ν be linearly independent.
   According to Lemma 2, there is $\xi\in\mathsf{E}_d$ such that
   $\langle\,\mu,\xi\,\rangle=1$, $\langle\,\nu,\xi\,\rangle=1$.
   Then,

$$\begin{aligned}
\nabla\kappa_1(\xi)&=-\frac{1}{\sqrt{\langle\,\xi,\Sigma\xi\,\rangle}}\mu,\\
\nabla\kappa_2(\xi)&=-\frac{1}{\sqrt{\langle\,\xi,\Gamma\xi\,\rangle}}\nu.
\end{aligned}$$

   The gradients are linearly independent, hence according to lemma 4, the family $\widetilde{\mathcal{K}}_{\mathcal{M}}$ is not uniformly monotone.

(b) Let $x \in \mathsf{E}_d$ such that Σx, Γx be linearly independent.
For $t > 0$ consider

$$\begin{aligned}\nabla\kappa_1(tx) &= -\frac{1}{t\sqrt{\langle x, \Sigma x\rangle}}\mu + \frac{\langle x, \mu\rangle}{t\langle x, \Sigma x\rangle^{3/2}}\Sigma x - \frac{1}{t^2\langle x, \Sigma x\rangle^{3/2}}\Sigma x,\\ \nabla\kappa_2(tx) &= -\frac{1}{t\sqrt{\langle x, \Gamma x\rangle}}\nu + \frac{\langle x, \nu\rangle}{t\langle x, \Gamma x\rangle^{3/2}}\Gamma x - \frac{1}{t^2\langle x, \Gamma x\rangle^{3/2}}\Gamma x.\end{aligned}$$

Then, gradients $\nabla\kappa_1(tx)$, $\nabla\kappa_2(tx)$ are linearly independent for $t > 0$ sufficiently small, hence according to lemma 4, family $\widetilde{\mathcal{K}}_{\mathcal{M}}$ is not uniformly monotone.

(c) Let μ, ν be linearly dependent and for all $x \in \mathsf{E}_d$ vectors Σx, Γx be linearly dependent.
Then, there is $m \in \mathbb{R}^d$ and $\alpha, \beta \in \mathbb{R}$ such that $\mu = \alpha m$, $\nu = \beta m$.
According to lemma 3, there is $\gamma > 0$ with property $\Gamma = \gamma\Sigma$.
Comparing largest eigenvalues of matrices, we receive $\gamma = 1$ and $\Gamma = \Sigma$.

Proposition of Lemma 6 is shown.

Q.E.D.

In accordance with Theorem 3, we derived family $\mathcal{K}_{\mathcal{M}}$ is uniformly monotone iff its functions can be expressed like $\kappa(\bullet|\alpha, \mu, \Sigma) = \phi_{\alpha,\Sigma} \circ \psi$. where

$$\begin{aligned}\phi_{\alpha,\Sigma} : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R} &\mapsto \frac{\alpha\sqrt{\Lambda(\mathbf{V})}}{\sqrt{\Lambda(\Sigma)}}t,\\ \psi : \mathsf{E}_d \to \mathbb{R} : x \in \mathbb{R}^d &\mapsto \frac{1 - \langle x, m\rangle}{\sqrt{\langle x, \mathbf{V}x\rangle}}.\end{aligned}$$

6.3 Probabilistic constraints

Let us denote $\mathsf{E}_d = \mathbb{R}^d \setminus \{0\}$ and define functions

$$\mathsf{h}(\bullet|\alpha, \mu, \Sigma) : \mathbb{R}^d \to \mathbb{R} : x \in \mathbb{R}^d \mapsto \mathsf{Prob}(\langle x, \mathsf{X}\rangle \le \alpha).,$$

whenever $\alpha > 0$, $\mu \in \mathbb{R}^d$, $\Sigma \in \mathsf{PDM}(d)$ and $\mathsf{X} \sim \mathrm{N}(\mu, \Sigma)$.

Lemma 7 *Let $\mathcal{M} \subset \mathbb{R}_+ \times \mathbb{R}^d \times \mathsf{PDM}(d)$. Then, a family of curves $\mathcal{H}_{\mathcal{M}} = \{\mathsf{h}(\bullet|\alpha, \mu, \Sigma) : (\alpha, \mu, \Sigma) \in \mathcal{M}\}$ is uniformly monotone iff there are $m \in \mathbb{R}^d$ and $\mathbf{V} \in \mathsf{PDM}(d)$ such that for all $(\alpha, \mu, \Sigma) \in \mathcal{M}$ we have $\mu = \alpha m$, $\Sigma = \frac{\Lambda(\Sigma)}{\Lambda(\mathbf{V})}\mathbf{V}$.*

Proof: Always, $\mathsf{h}(\mathbf{0}|\alpha, \mu, \Sigma) = 1$. Unity dominates any probability.
Therefore, $\mathcal{H}_{\mathcal{M}}$ is uniformly monotone iff $\widetilde{\mathcal{H}}_{\mathcal{M}} = \left\{\widetilde{\mathsf{h}}(\bullet|\alpha, \mu, \Sigma) : (\alpha, \mu, \Sigma) \in \mathcal{M}\right\}$ is uniformly monotone, where $\widetilde{\mathsf{h}}$ is the restriction of h to E_d.

For all $x \in \mathsf{E}_d$

$$\widetilde{\mathsf{h}}(x|\alpha, \mu, \Sigma) = \Phi\left(\frac{\alpha - \langle x, \mu\rangle}{\sqrt{\langle x, \Sigma x\rangle}}\right).$$

Probability distribution function Φ of the standard Gaussian variable is increasing, therefore, $\widetilde{\mathcal{H}}_{\mathcal{M}}$ is uniformly monotone iff $\mathcal{K}_{\mathcal{M}}$ is uniformly monotone; which is defined in Lemma 6,

Q.E.D.

In accordance with Theorem 3, we derived family $\mathcal{H}_{\mathcal{M}}$ is uniformly monotone iff its functions can be expressed like $\widetilde{\mathsf{h}}\left(\bullet|\alpha,\mu,\Sigma\right) = \phi_{\alpha,\Sigma} \circ \psi$. where

$$\phi_{\alpha,\Sigma} : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R} \mapsto \Phi\left(\frac{\alpha\sqrt{\Lambda\left(\mathbf{V}\right)}}{\sqrt{\Lambda\left(\Sigma\right)}}t\right),$$

$$\psi : \mathsf{E}_d \to \mathbb{R} : x \in \mathbb{R}^d \mapsto \frac{1 - \langle\, x, m\,\rangle}{\sqrt{\langle\, x, \mathbf{V}x\,\rangle}}.$$

Now, we proceed to the example stated in [7]; i.e. we focus our interest to uniformly quasi-concave families. Unfortunately, considered function are not quasi-concave on whole $\mathbb{R}^d$. We have to restrict their definition region to a convenient $E \subset \mathbb{R}^d$ and denote

$$\mathsf{q}\left(\bullet|\alpha,\mu,\Sigma,E\right) : E \to \mathbb{R} : x \in E \mapsto \mathsf{Prob}\left(\langle\, x, \mathsf{X}\,\rangle \le \alpha\right).,$$

Lemma 8 *Let* $\mathcal{M} \subset \mathbb{R}_+ \times \mathbb{R}^d \times \mathsf{PDM}\left(d\right)$ *and* $E \subset \mathbb{R}^d$. *If there are* $m \in \mathbb{R}^d$ *and* $\mathbf{V} \in \mathsf{PDM}\left(d\right)$ *such that for all* $(\alpha,\mu,\Sigma) \in \mathcal{M}$ *we have* $\mu = \alpha m$, $\Sigma = \frac{\Lambda(\Sigma)}{\Lambda(\mathbf{V})}\mathbf{V}$ *and* $\mathsf{q}\left(\bullet|\alpha,\mu,\Sigma,E\right)$ *is quasi-concave then the family of curves* $\mathit{\Pi}_{\mathcal{M}} = \left\{\mathsf{q}\left(\bullet|\alpha,\mu,\Sigma,E\right) \,:\, (\alpha,\mu,\Sigma) \in \mathcal{M}\right\}$ *is uniformly quasi-concave.*

Proof: Family $\mathit{\Pi}_{\mathcal{M}}$ is uniformly monotone, since family $\mathcal{H}_{\mathcal{M}}$ is uniformly monotone, according to Lemma 7.

Functions of $\mathit{\Pi}_{\mathcal{M}}$ are assumed to be quasi-concave.

Consequently, $\mathit{\Pi}_{\mathcal{M}}$ is uniformly quasi-concave family, see Definition 6.

Q.E.D.

Appropriate choice of definition region is, for example,

$$E \quad = \bigcap_{(\alpha,\mu,\Sigma)\in\mathcal{M}} \left\{x \in \mathbb{R}^d \,:\, \mathsf{h}\left(x|\alpha,\mu,\Sigma\right) \ge \frac{1}{2}\right\}, \tag{1}$$

see Lemma 2.2 of [7].

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